Basic Trigonometry

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Trigonometry

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For Missy,

the flower of Lackawanna.
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Chapter 1

Right-Triangle Trigonometry

1.1 Introduction

In this section we will recall some theorems on triangles from geometry.

Theorem 1.1.1. The sum of the angles in any triangle is 180°

In terms of Figure 1.1.1: \( m\angle A + m\angle B + m\angle C = 180° \).

Definition 1.1.2. Two triangles are similar, denoted \( \triangle ABC \sim \triangle DEF \) if the measures of corresponding angles are equal and if the ratios of the lengths of corresponding sides are equal:

\[
\frac{a}{d} = \frac{b}{e} = \frac{c}{f}
\]
Theorem 1.1.3. (AA Similarity) If two pairs of angles in two triangles are congruent, then the triangles are similar.

In terms of Figure 1.1.2: if $m\angle A = m\angle D$ and $m\angle B = m\angle E$, then $\triangle ABC \sim \triangle DEF$.

Recall that if two triangles are similar, then ratios of pairs of corresponding sides are equal. Again, in terms of Figure 1.1.2: if $\triangle ABC \sim \triangle DEF$, then (among other ratios)

$$\frac{b}{c} = \frac{e}{f} \quad \text{and} \quad \frac{b}{e} = \frac{a}{d}$$

Conventions: Note that in Figures 1.1.1 and 1.1.2 the angles of the two triangles are labeled by upper case letters and the sides opposite the angles are labeled with the corresponding lower case letter. We will follow this convention throughout these notes. In geometry angles, which are geometrical objects, are rigorously distinguished from their measures, which are numbers. In these notes we will allow not-technically-correct identification of an angle with its measure. Thus we will write $A = 63^\circ$ in place of $m\angle A = 63^\circ$.

Theorem 1.1.4. (Pythagoras)

In a right triangle with side lengths $a$, $b$, and $c$, where $c$ is the length of the hypotenuse,

$$c^2 = a^2 + b^2$$

Figure 1.1.3
Section 1.1 Problems

For Problems 1–4 we are given that $\triangle ABC \sim \triangle DEF$.

1. If $a = 2$ and $b = 3$, find $\frac{d}{e}$.
2. If $a = 3$, $b = 4$, and $d = 5$, find $e$.
3. If $a = 2$ and $c = 5$, find $\frac{f}{a}$.
4. If $b = 4$, $c = 7$, and $d = 5$, find $f$.

For Problems 5–8 we are given that $\triangle ABC$ is a right triangle with $C$ a right angle.

5. If $a = 2$ and $b = 3$, find $c$.
6. If $a = 3$ and $b = 5$, find $b$.
7. If $a = \sqrt{2}$ and $b = \sqrt{3}$, find $c$.
8. If $a = 1$ and $c = 2$, find $b$.

9. In $\triangle ABC$, if $A = 23^\circ$ and $B = 55^\circ$ find $C$.
10. In $\triangle DEF$, if $D = 33^\circ$ and $E = 47^\circ$ find $F$.
11. In $\triangle XYZ$, if $X = Y$ and $Z = 100^\circ$ find $X$.
12. In $\triangle PQR$, if $P = 2Q$ and $R = 120^\circ$ find $P$.
13. If $\triangle ABC$ with $C$ a right angle and $A = 27^\circ$, find $B$.
14. If $\triangle ABC$ with $C$ a right angle and $B = 62^\circ$, find $A$. 
1.2 Right-Triangle Trigonometry

Consider a right triangle that contains an angle with measure 27°.

As usual, the side opposite the right angle is called the hypotenuse of the triangle. The side that forms the 27° angle with the hypotenuse is called the adjacent side to the angle and the last side is called the opposite side of the angle. Any right triangle containing a 27° angle will be similar to the one in Figure 1.2.2.

In any right triangle containing an angle of measure 27°, the ratios \( \frac{\text{opp}}{\text{hyp}} \), \( \frac{\text{adj}}{\text{hyp}} \), and \( \frac{\text{opp}}{\text{adj}} \) will be the same. These ratios are called the sine, cosine and tangent, respectively.

The same can be said of any acute angle. This leads to the following definition.

**Definition 1.2.1.** Let \( \theta \) be an acute angle in a right triangle. We define the following functions on \( \theta \).

\[
\sin \theta = \frac{\text{opp}}{\text{hyp}}, \\
\cos \theta = \frac{\text{adj}}{\text{hyp}}, \\
\tan \theta = \frac{\text{opp}}{\text{adj}}
\]

In days gone by, the values of the trigonometric functions were tabulated and books on trigonometry contained tables such as Table 1.2.1. Today, of course, these values are easily computed on a scientific calculator.
To \textbf{solve a triangle} means to find the measures of unknown angles and lengths of unknown sides of a given triangle.

Example 1.2.2.

Solve the triangle given in the diagram.

Since the acute angles in a right triangle are complementary,

\[ B = 90^\circ - 34^\circ = 56^\circ \]

\[ \tan 34^\circ = \frac{a}{3.5} \]

\[ a = 3.5 \tan 34^\circ \]

\[ = 2.4 \]

Using the Pythagoras theorem:

\[ c^2 = a^2 + b^2 \]
\[ c^2 = 2.4^2 + 3.5^2 \]
\[ c = \sqrt{2.4^2 + 3.5^2} \]
\[ = 4.2 \]

The values in Table 1.2.1 are decimal approximations of the values of the trigonometric functions. If the exact value of one of the trigonometric functions happens to be known, then the exact value of the others can be found.
Example 1.2.3.

Given that for acute angle $\theta$, $\sin \theta = \frac{3}{7}$, find $\cos \theta$ and $\tan \theta$.

Choose a right triangle containing $\theta$ with opposite side of length 3 and hypotenuse of length 7. Let $x$ be the length of the remaining side.

Using the Pythagorean theorem,

\[
x^2 + 3^2 = 7^2
\]

\[
x^2 = 40
\]

\[
x = \sqrt{40} = 2\sqrt{10}
\]

\[
\cos \theta = \frac{2\sqrt{10}}{7}
\]

\[
\tan \theta = \frac{3}{2\sqrt{10}}
\]

One of the earliest uses of trigonometry is in indirect measurement problems. The height of a tall building would be difficult to measure directly. It is easier to measure a horizontal distance along the ground and the angle of a sightline and then use trigonometry to deduce the desired height.

Example 1.2.4.

At a point 105 feet away from a building the angle between the ground the top of the building is 55°. Find the height of the building.

Let $h$ be the height of the building. Then,

\[
\tan 55^\circ = \frac{h}{105}
\]

\[
h = 105 \tan 55^\circ
\]

\[
h = 150.0
\]

There are three remaining trigonometric functions: cosecant, secant, cotangent, which are simply the reciprocals of sine, cosine, and tangent, respectively.
**Definition 1.2.5.** For an angle $\theta$ we define,

\[
csc \theta = \frac{1}{\sin \theta} \\
sec \theta = \frac{1}{\cos \theta} \\
cot \theta = \frac{1}{\tan \theta}
\]

**Example 1.2.6.**

Given that for an acute angle $\theta$, $\cos \theta = \frac{4}{5}$, find the values of the remaining 5 trigonometric functions on $\theta$.

Choose a right triangle containing $\theta$ with adjacent side of length 4 and hypotenuse of length 5. Let $y$ be the length of the remaining side.

Using the Pythagoras theorem,

\[
y^2 + 4^2 = 5^2 \\
y^2 = 9 \\
y = 3
\]

\[
\sin \theta = \frac{3}{5} \\
csc \theta = \frac{5}{3} \\
\cos \theta = \frac{4}{5} \\
sec \theta = \frac{5}{4} \\
\tan \theta = \frac{3}{4} \\
cot \theta = \frac{4}{3}
\]

### The Inverse Trigonometric Functions

Given an acute angle $\theta$, the trigonometric functions assign a number to that angle. The inverse trigonometric functions reverse that process. That is given the value of a trigonometric function on some angle, the inverse trigonometric function will give the measure of the angle. The inverse trigonometric functions are denoted: $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$. Scientific calculators have buttons for these functions.

For example, using a scientific calculator,

\[
\sin 52^\circ = 0.788010754 \\
\sin^{-1} 0.788010754 = 52^\circ
\]
Example 1.2.7.\[\triangle ABC\] is a right triangle with \(C\) a right angle. Given that \(a = 5.1\) and \(c = 8.4\), solve \(\triangle ABC\).

Using the Pythagoras theorem, \(b = 6.7\).

\[
\begin{align*}
sin A &= \frac{5.1}{8.4} \\
A &= \sin^{-1}\left(\frac{5.1}{8.4}\right) \\
A &= 37.4^\circ \\
B &= 90^\circ - 37.4^\circ = 52.6^\circ
\end{align*}
\]

Special Triangles

In most cases evaluating the trigonometric functions on a given angle \(\theta\) is a calculator problem. There are three acute angles, 30°, 45°, and 60°, on which the exact value of the trigonometric angles can be found.

Consider an equilateral triangle with side length 2. An altitude drawn from one of the angles bisects the angle and the opposite side.

Using the Pythagoras theorem,

\[
\begin{align*}
1^2 + h^2 &= 2^2 \\
h^2 &= 3 \\
h &= \sqrt{3}
\end{align*}
\]

We now construct a triangle as follows: choose two perpendicular lines and construct a segment of length 1 on each line, each with the point of intersection for one endpoint.
CHAPTER 1. RIGHT-TRIANGLE TRIGONOMETRY

Using the Pythagoras theorem,

\[ a^2 + b^2 = c^2 \]

\[ c^2 = 2 \]

\[ c = \sqrt{2} \]

We can now use the following triangles to evaluate the trigonometric functions on the angles, 30°, 45°, and 60°.

Section 1.2 Problems

For problems 1–9 \( \Delta ABC \) is a right triangle with \( C \) a right angle. Solve the triangle with the given information. Give your answer correct to one decimal place.

1. \( a = 8.7; A = 22^\circ \)
2. \( b = 3.1; A = 39^\circ \)
3. \( c = 8.1; B = 52^\circ \)
4. \( a = 3.3; b = 4.5 \)
5. \( a = 4.3; c = 7.6 \)
6. \( b = 5.8; c = 8.5 \)
7. \( b = 7.9; B = 47^\circ \)
8. \( c = 9.1; A = 17^\circ \)
9. \( c = 5.5; B = 55^\circ \)
For problems 10–15 $\triangle ABC$ is a right triangle with $C$ a right angle. Given the value of one of the trigonometric function, find the values of the remaining 5. Give exact-value answers.

10. \(\sin \theta = \frac{3}{7}\).
11. \(\cos \theta = \frac{5}{13}\).
12. \(\tan \theta = \frac{5}{2}\).
13. \(\sin \theta = \frac{1}{2}\).
14. \(\cos \theta = \frac{\sqrt{2}}{5}\).
15. \(\tan \theta = \sqrt{3}\).

16. Using Figure 1.2.3, evaluate the six trigonometric functions on
   (a) \(\theta = 30^\circ\)
   (b) \(\theta = 45^\circ\)
   (c) \(\theta = 60^\circ\)

In application problems, the **angle of depression** is the angle between the line of sight of an observer looking downward and the horizontal. The **angle of elevation** is the angle between the line of sight of an upward looking observer and the horizontal.

17. Suzy’s eyes are 5 feet above the ground. She is standing 12 feet away from the base of a flag pole and the angle of elevation from her eyes to the top of the pole is 23°. What is the height of the flag pole?

18. An observer on the ground measures the angle of elevation of an airplane to be 38°. A second observer, standing 4.3 miles from the first observer, is directly below the plane. What is the altitude of the plane?

19. Find the angle of elevation of the top of a 63-foot-tall building from a point on the ground that is 97 feet from the base of the building.

20. In the previous problem find the angle of elevation of the top of the building from the eye level of an observer whose eyes are at a height of 5.4 feet. (The observer stands 97 feet from the base of the building.)

21. The angle of elevation to the top of an oak tree from a point on the ground 59 feet from the base of the tree is 43°. What is the distance from the point to the top of the tree?
Evel Knievel was a daredevil who rode his motorcycle over ramps to jump various objects. He estimates that, in order to be safe, the angle of the ramp with the ground should be no more than $17^\circ$.

22. Evil Knievel is going to jump over 5 school buses that have a height of 9 feet. If his ramp has an angle of $17^\circ$, what is the distance along the ground from the base of the ramp to the first school bus?

23. In order to jump over a 12-foot-high pen containing mountain lions, Evel Knievel constructs a ramp that measures 50 feet along the ground. What is the angle that the ramp makes with the ground?

24. Evel Knievel is going to construct a ramp to jump over some locomotives that have a height 14 feet. If the angle of the ramp with the ground is $17^\circ$, what is the shortest ramp (not the distance along the ground) that our daredevil can construct?
1.3 Trigonometric Functions of any Angle

The trigonometric functions defined in Section 1.2 can only be evaluated on acute angles. In this section we expand the definitions so that the trigonometric functions can be evaluated on any angle.

Angles in Standard Position

Recall the following definition from geometry.

Definition 1.3.1.

An angle is the union of two rays with a common endpoint.

We will adopt a more dynamic view of angles. An angle in standard position is angle in the coordinate plane with its vertex at the origin. The angle is formed by placing both rays along the positive $x$-axis, fixing one in place and rotating the other. The fixed ray is called the initial ray and the rotated ray is called the terminal ray. In forming such an angle the terminal ray may be rotated either in the counterclockwise or clockwise directions. They are distinguished by defining their measures to be positive or negative, respectively.

![Figure 1.3.1: Angles in Standard Position](image)

Angles formed by counterclockwise rotation have positive measures.

Angles formed by clockwise rotation have negative measures.
Recall that the four quadrants of the coordinate plane are defined as in Figure 1.3.2.

Note the inequalities defining the quadrants are strict, so that the coordinate axes from the boundaries of the quadrants, but do not belong to the quadrants.

Angles whose terminal rays lie on the coordinate axes are called **boundary angles**

---

0° This is a nonstandard usage. Many authors call these angles quadrantal angles. We have adopted the term boundary angle as more descriptive. The terminal ray of such an angle lies in the boundary of the quadrants, not in the quadrants.
Definition 1.3.2.

For an angle, $\theta$, in standard position whose terminal ray is interior to one of the quadrants, the reference angle, $\theta'$, of $\theta$ is the acute angle formed by the terminal ray and the $x$-axis.

Example 1.3.3.

Find the reference angle for the following angles: $150^\circ$, $-130^\circ$, $300^\circ$

Angles in standard position are formed by rotation of the terminal ray. There is no reason to restrict the rotation to once-round. This can result in different angles having the same terminal ray.
Definition 1.3.4.

Angles with the same terminal ray are called \textit{coterminal} angles.

Example 1.3.5.

Find three angles coterminal with $\theta = 30^\circ$.

(i) $30^\circ + 360^\circ = 390^\circ$

(ii) $30^\circ + 2 \cdot 360^\circ = 750^\circ$

(iii) $30^\circ - 360^\circ = -330^\circ$
The Definition of the Trigonometric Functions

We now turn to the definition of the trigonometric functions on angles in standard position. Consider first an angle in standard position whose terminal ray lies in the first quadrant. Let \((x, y)\) be a point on the terminal ray other than the origin. Form a triangle by dropping a perpendicular from this point to the \(x\)-axis. This is called the reference triangle if \(\theta\). Let \(r\) be the distance from the point to the origin.

By the Pythagoras theorem

\[ r^2 = x^2 + y^2 \]

Using the definitions of the trigonometric functions of an acute angle,

\[
\begin{align*}
\sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\
\cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\
\tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y}
\end{align*}
\]

We now make the observation that the values of the trigonometric functions given above do not depend on the terminal ray of the angle lying in the first quadrant, so we take them to be our general definition.

**Definition 1.3.6.** (The trigonometric functions.)

Let \(\theta\) be an angle in standard position, \(P(x, y)\) be a point on the terminal ray of \(\theta\), and let \(r\) be the distance from \(P\) to the origin. We define the values of the trigonometric functions on \(\theta\) as follows:

\[
\begin{align*}
\sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\
\cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\
\tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y}
\end{align*}
\]
In Definition 1.3.6, \( r \) is a distance, so \( r > 0 \). However, \((x, y)\) are the coordinates of a point and so they may be positive, negative, or zero. Regardless of the values of \( x \) and \( y \), \( r \) still satisfies the Pythagoras relation: \( r^2 = x^2 + y^2 \).

**Example 1.3.7.**

Let \( \theta \) be an angle in standard position whose terminal ray passes through the point \((2, -5)\). Find the values of the six trigonometric functions on \( \theta \).

\[
\begin{align*}
r^2 &= x^2 + y^2 \\
r^2 &= 2^2 + (-5)^2 \\
r &= \sqrt{29} \\
\sin \theta &= -\frac{5}{\sqrt{29}} & \csc \theta &= -\frac{\sqrt{29}}{5} \\
\cos \theta &= \frac{2}{\sqrt{29}} & \sec \theta &= \frac{\sqrt{29}}{2} \\
\tan \theta &= -\frac{5}{2} & \cot \theta &= -\frac{2}{5}
\end{align*}
\]

As Example 1.3.11 shows, the values of the trigonometric functions can sometimes be negative. The signs of the trigonometric functions depend on the signs of the coordinates \((x, y)\). The diagram to the left is helpful in remembering which functions are positive in which quadrants.

Figure 1.3.4
Example 1.3.8.

Given that \( \sin \theta = \frac{3}{4} \) and \( 90^\circ < \theta < 180^\circ \) Find the values of the remaining five trigonometric functions on \( \theta \).

Since \( \sin \theta = \frac{3}{4} > 0 \), the terminal ray of \( \theta \) must lie in Quadrants I or II. The fact that \( 90^\circ < \theta < 180^\circ \) implies that the terminal ray of \( \theta \) lies in Quadrant II.

Form the reference triangle of \( \theta \) using the point \((x, 3)\).

\[
\begin{align*}
\sin \theta &= \frac{3}{4} \\
\csc \theta &= \frac{4}{3} \\
\cos \theta &= -\frac{\sqrt{7}}{4} \\
\sec \theta &= -\frac{4}{\sqrt{7}} \\
\tan \theta &= -\frac{3}{\sqrt{7}} \\
\cot \theta &= -\frac{\sqrt{7}}{3}
\end{align*}
\]

In this case, the solution we are looking for is \( x = -\sqrt{7} \).

Finally we observe that for any angle \( \theta \) interior to one of the quadrants, \( \theta' \) is an ordinary acute angle and,

\[
\begin{align*}
\sin \theta' &= \frac{|y|}{r} \\
\csc \theta' &= \frac{r}{|y|} \\
\cos \theta' &= \frac{|x|}{r} \\
\sec \theta' &= \frac{r}{|x|} \\
\tan \theta' &= \frac{|y|}{|x|} \\
\cot \theta' &= \frac{|x|}{|y|}
\end{align*}
\]

Hence the value of a trigonometric function on \( \theta' \) is the absolute value of the value of the function on \( \theta \).
CHAPTER 1. RIGHT-TRIANGLE TRIGONOMETRY

Theorem 1.3.9. If \( \theta \) is an angle in standard position whose terminal ray lies interior to one of the quadrants, then

\[
\begin{align*}
\sin \theta' &= |\sin \theta| & \csc \theta' &= |\csc \theta| \\
\cos \theta' &= |\cos \theta| & \sec \theta' &= |\sec \theta| \\
\tan \theta' &= |\tan \theta| & \cot \theta' &= |\cot \theta|
\end{align*}
\]

Example 1.3.10.

Find the values of \( \sin 150^\circ \), \( \cos 150^\circ \), and \( \tan 150^\circ \).

The reference angle of \( \theta = 150^\circ \) is \( \theta' = 30^\circ \).

Using Theorem 1.3.9 and Figure 1.3.4

\[
\begin{align*}
\sin 30^\circ &= \frac{1}{2} \\
\cos 30^\circ &= \frac{\sqrt{3}}{2} \\
\tan 30^\circ &= \frac{1}{\sqrt{3}}
\end{align*}
\]

\[
\begin{align*}
\sin 150^\circ &= +\frac{1}{2} \\
\cos 150^\circ &= -\frac{\sqrt{3}}{2} \\
\tan 150^\circ &= -\frac{1}{\sqrt{3}}
\end{align*}
\]

Because it is easy to find points on the coordinate axes, we can directly use the definition of the trigonometric functions to evaluate on the boundary angles.
Example 1.3.11.

Find the values of \( \sin 90^\circ \), \( \cos 90^\circ \), and \( \tan 90^\circ \).

The point \( P(0, 1) \) lies on the terminal ray of the \( 90^\circ \) angle and has distance \( r = 1 \) to the origin.

\[
\begin{align*}
\sin 90^\circ &= \frac{y}{r} = \frac{1}{1} = 1 \\
\cos 90^\circ &= \frac{x}{r} = \frac{0}{1} = 0 \\
\tan 90^\circ &= \frac{y}{x} = \frac{1}{0} \text{ (undefined)}
\end{align*}
\]

Section 1.3 Problems

For problems 1–9, sketch the given angle in standard position and find its reference angle.

1. \( \theta = 120^\circ \). 
2. \( \theta = 200^\circ \). 
3. \( \theta = 295^\circ \). 
4. \( \theta = -220^\circ \). 
5. \( \theta = 27^\circ \). 
6. \( \theta = -55^\circ \). 
7. \( \theta = 135^\circ \). 
8. \( \theta = 240^\circ \). 
9. \( \theta = 310^\circ \). 

For problems 10–15, find three angles coterminal to the given angle.

10. \( \theta = 120^\circ \). 
11. \( \theta = 200^\circ \). 
12. \( \theta = 295^\circ \). 
13. \( \theta = -220^\circ \). 
14. \( \theta = 27^\circ \). 
15. \( \theta = 310^\circ \). 

For problems 16–21, evaluate the six trigonometric functions on the angle \( \theta \) whose terminal ray passes through the given point. Give exact-value answers.

16. \( P(-2, 5) \). 
17. \( P(-3, -4) \). 
18. \( P(5, -3) \). 
19. \( P(-1, 0) \). 
20. \( P(2, 4) \). 
21. \( (0, -1) \).

For problems 22–27, evaluate the six trigonometric functions on the given angle. Give exact-value answers.

22. \( \theta = 120^\circ \). 
23. \( \theta = 225^\circ \). 
24. \( \theta = 330^\circ \). 
25. \( \theta = 180^\circ \). 
26. \( \theta = 240^\circ \). 
27. \( \theta = 270^\circ \).
Chapter 2

Trigonometric Identities

2.1 Basic Trigonometric Identities

While there are six trigonometric functions, they are not independent. In this section we will explore some relationships among them. We will use the right-triangle definition of the trigonometric functions for geometric motivation, but the identities we find will hold for the generalized definitions as well.

The Reciprocal Identities

This first set of identities is simply a restatement of Definition 1.2.5:

\[
\begin{align*}
\sec \theta &= \frac{1}{\cos \theta} & \cos \theta &= \frac{1}{\sec \theta} \\
\csc \theta &= \frac{1}{\sin \theta} & \sin \theta &= \frac{1}{\csc \theta} \\
\cot \theta &= \frac{1}{\tan \theta} & \tan \theta &= \frac{1}{\cot \theta}
\end{align*}
\]

The trigonometric ratios depend on the fact that any two right triangles containing the acute angle \( \theta \) must be similar. It is also true that any individual right triangle that contains the angle acute \( \theta \) must satisfy the Pythagoras theorem. The next two sets of identities establish relationships among the the trigonometric ratios derived from similarity relations and the Pythagoras theorem.

The Quotient Identities

This set of identities establishes relationships among ratios of trigonometric functions.
CHAPTER 2. TRIGONOMETRIC IDENTITIES

Referring to Figure 2.1.1,

\[
\begin{align*}
\frac{\sin \theta}{\cos \theta} &= \frac{b/c}{a/c} = \frac{b}{c} \cdot \frac{a}{a} = \frac{b}{a} = \tan \theta \\
\frac{\cos \theta}{\sin \theta} &= \frac{a/c}{b/c} = \frac{a}{b} = \cot \theta
\end{align*}
\]

Quotient Identities

\[
\frac{\sin \theta}{\cos \theta} = \tan \theta \quad \frac{\cos \theta}{\sin \theta} = \cot \theta
\]

The Pythagoras Identities

The right triangle in Figure 2.1.1 satisfies the Pythagoras theorem: \( c^2 = a^2 + b^2 \). We can use this theorem to investigate relationships among the squares of the quotients that define the trigonometric functions. First we will compute the sum of the squares of the sine and cosine ratios.\(^1\)

\[
\cos^2 \theta + \sin^2 \theta = \left( \frac{a}{c} \right)^2 + \left( \frac{b}{c} \right)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2}
\]

\[
= \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1
\]

Putting all this together, we have:

\[
\cos^2 \theta + \sin^2 \theta = 1 \quad (\dagger)
\]

\(^1\)Note that \(\sin^2 \theta\) is the notation for \((\sin \theta)^2\) and similarly for the other trigonometric functions.
We can now use identity (†) together with the quotient and reciprocal identities to generate two more identities.

Dividing identity (†) by \( \cos \theta \):

\[
\begin{align*}
\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} &= \frac{1}{\cos^2 \theta} \\
1 + \tan^2 \theta &= \sec^2 \theta
\end{align*}
\]

Dividing identity (†) by \( \sin \theta \):

\[
\begin{align*}
\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} &= \frac{1}{\sin^2 \theta} \\
\frac{\cos^2 \theta}{\sin^2 \theta} + 1 &= \csc^2 \theta
\end{align*}
\]

\[
\text{Pythagoras Identities}
\]

\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 \\
1 + \tan^2 \theta &= \sec^2 \theta \\
1 + \cot^2 \theta &= \csc^2 \theta
\end{align*}
\]

Using the basic identities established so far and some algebra, we can establish other identities. The process of proving a trigonometric identity is one of simplification. Pick one side of the identity and use algebra and the basic identities given above to simplify it. The end of the process should yield the other side of the identity.

**Example 2.1.1.**

Prove the identity: \( \cos \theta + \sin \theta \tan \theta = \sec \theta \).

\[
\begin{align*}
\cos \theta + \sin \theta \tan \theta &= \cos \theta + \frac{\sin \theta}{\cos \theta} \\
&= \frac{\cos \theta}{\cos \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} \\
&= \frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta} \\
&= \frac{1}{\cos \theta} \\
&= \sec \theta
\end{align*}
\]
Example 2.1.2.
Prove the identity: \( \cot \theta \csc \theta = \cos \theta \).

\[
\cot \theta \csc \theta = \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\sin \theta} = \frac{\cos \theta \sin \theta}{\sin \theta} = \cos \theta
\]

Example 2.1.3.
Prove the identity: \( \tan \theta + \cot \theta = \csc \theta \sec \theta \).

\[
\tan \theta + \cot \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta} = \frac{1}{\sin \theta} \cdot \frac{1}{\cos \theta} = \csc \theta \sec \theta
\]

Section 2.1 Problems

For problems 1–10, prove the trigonometric identity.

1. \( \frac{\tan \theta}{\sin \theta} = \sec \theta \).
2. \( \frac{\tan \theta}{\sec \theta} = \sin \theta \).
3. \( \sin \theta + \sin \theta \cot^2 \theta = \csc \theta \).
4. \( \sin \theta \cos \theta \tan \theta = 1 = \cos^2 \theta \).
5. \( \tan \theta \cos \theta = \sin \theta \).
6. \( \sin \theta \csc^2 \theta = \csc \theta \).
7. \( \frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} = \frac{1}{\cos^2 \theta - \sin^2 \theta} \).
8. \( \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta \).
9. \( \frac{\sec \theta - 1}{\sec \theta + 1} = \frac{1 - \cos \theta}{1 + \cos \theta} \).
10. \( \tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta \).
Midcourse Review

Review

1. Find the values of all six trigonometric functions on angle \( A \) in the triangle below.

![Diagram of a triangle with sides labeled a, b, and c, and angle A labeled.]  

2. Find the reference angles for each of the following angles.

(a) \( 120^\circ \)  \hspace{1cm}  (c) \( 350^\circ \)  \hspace{1cm}  (e) \( -330^\circ \)

(b) \( 212^\circ \)  \hspace{1cm}  (d) \( -97^\circ \)  \hspace{1cm}  (f) \( -45^\circ \)

3. Given that \( \alpha \) is an acute angle and sin \( \alpha = \frac{2}{3} \), find the values of the remaining 5 trigonometric functions on \( \theta \).

4. Given that \( 90^\circ < \theta < 180^\circ \) and sin \( \theta = \frac{2}{3} \), find the values of the remaining 5 trigonometric functions on \( \theta \).

5. Solve the right triangle in the figure below.

![Diagram of a right triangle with sides labeled c, 3.1, and 4.7, and angle C labeled 33°.]  

6. Evaluate \( \sin^2 23^\circ + \cos^2 23^\circ \).

7. Given that \( \sin 23^\circ = 0.3907 \), find:

(a) \( \sin 157^\circ \)  \hspace{1cm}  (d) \( \cos 67^\circ \)

(b) \( \sin 203^\circ \)  \hspace{1cm}  (e) \( \cos 113^\circ \)

(c) \( \sin 337^\circ \)  \hspace{1cm}  (f) \( \cos(-113^\circ) \)

8. Use the special triangles to evaluate:

(a) \( \cos 120^\circ \)  \hspace{1cm}  (d) \( \cos 135^\circ \)

(b) \( \tan 210^\circ \)  \hspace{1cm}  (e) \( \tan(-60^\circ) \)

(c) \( \sin 330^\circ \)  \hspace{1cm}  (f) \( \sin 150^\circ \)

9. Evaluate:

(a) \( \cos 90^\circ \)  \hspace{1cm}  (d) \( \sin 270^\circ \)

(b) \( \tan 180^\circ \)  \hspace{1cm}  (e) \( \tan 270^\circ \)

(c) \( \sin 0^\circ \)  \hspace{1cm}  (f) \( \cos 0^\circ \)

10. Given \( \sin \theta = -0.225 \).

(a) Find \( \theta' \), the reference angle for \( \theta \).

(b) Given that \( \theta \) lies in Quad. IV, evaluate:

i. \( \cos \theta \)

ii. \( \tan \theta \)

11. Solve the right triangle in the figure below.

![Diagram of a right triangle with sides labeled c, 3.1, and 4.7, and angle C labeled 33°.]  

12. Prove the following trigonometric identities:

(a) \( \cot \theta \ \csc \theta = \cos \theta \)

(b) \( \cot \alpha + \tan \alpha = \sec \alpha \ \csc \alpha \)

(c) \( \tan \beta \ \sin \beta + \cos \beta = \sec \beta \)

(d) \( \frac{1}{1 - \sin \theta} - \frac{1}{1 + \sin \theta} = 2 \tan \theta \ \sec \theta \)
CHAPTER 2. TRIGONOMETRIC IDENTITIES

Practice Test

1. If \( \theta \) is an acute angle and \( \sin \theta = \frac{3}{7} \), find \( \tan \theta \). Give an exact-value answer.

2. In the triangle below, find the length of side \( a \). Give your answer correct to one decimal place.

![Problem 2](image)

3. Given that \( \cos \theta = \frac{4}{5} \) and \( 270^\circ < \theta < 360^\circ \), find the exact value of \( \sin \theta \).

4. Find the reference angle for \( \theta = 265^\circ \).

5. Given that \( \sin \theta = -0.5731 \) and \( \theta \) lies in Quad IV. For this problem, give your answers correct to three digits after the decimal.
   (a) Find \( \theta' \), the reference angle for \( \theta \).
   (b) \( \cos \theta = \)
   (c) \( \tan \theta = \)

6. Given that \( \tan \theta = \frac{5}{\sqrt{2}} \), find the exact value of \( \cot(90^\circ - \theta) \).

7. Use your knowledge of the special triangles and the definition of the trigonometric functions to give the exact value of the following:

8. In the triangle below, find the measure of angle \( A \). Give your answer correct to two decimal places.

![Problem 8](image)

9. Find the value of the six trigonometric functions on the angle shown in the diagram below

![Problem 9](image)

10. Prove the following identity:
   \[ \csc \theta (\csc \theta - \sin \theta) = \cot^2 \theta. \]
Chapter 3

Radian Measure

3.1 The Definition of Radian Measure

Up to now we have been measuring angles in degrees without much comment. We now wish to step back and consider what is meant by degree measure. In order to define an angle measure, divide a circle into $2n$ equal parts by choosing $n$ equally spaced points on the top half of the circle and drawing diameters from those points. The result is that $2n$ equal central angles of the circle are formed. We then take the measure of one of these central angles to be the standard angle measure.

The measure of an angle is then the number of nonoverlapping standard angles that fit into the angle. Degree measure is the result of choosing to divide the circle into 360 equal parts. The choice of 360 is arbitrary. Another choice will result in an equally valid angle measure, rather like switching from feet to inches in linear measure. In contrast to linear measure, it is possible to construct an intrinsic or natural measure for angles, that is a measure that does not involve an arbitrary choice. That is the content of this section.
Definition 3.1.1. (Radian Measure)
Let \( \theta \) be an central angle of a circle or radius \( r \), and let \( s \) be the length of the arc subtended by \( \theta \).

The radian measure of \( \theta \) is given by:

\[
\theta = \frac{s}{r}
\]

It may seem that Definition 3.1.1 depends on the choice of a circle, but since all circles are similar, the defining ratio will remain the same from circle to circle. The definition is also reminiscent of the definition of \( \pi \). Recall that \( \pi \) is the ratio of the circumference to the diameter of a circle (Again, this does not depend on the circle chosen):

\[
\pi = \frac{c}{d}
\]

In fact, this observation will be the key to converting angle measures between degrees and radians.

Radian-Degree Conversions

If \( \theta = 180^\circ \), then the radian measure of \( \theta \) is given by

\[
\theta = \frac{s}{r} = \frac{c/2}{d/2} = \frac{c}{d} = \pi
\]

Hence,

\[180^\circ = \pi\]
We note that radian measure is actually unitless. The radius and arc length are measured in linear units, say inches. Then
\[ \theta = \frac{s \text{ in}}{r \text{ in}} = \frac{s}{r} \text{ (no units)} \]

Nevertheless, it is a useful mnemonic device to append rad to radian measure as if it were a unit. For the purposes of converting between the two angle measures it is also useful to append deg to an angle rather than the usual °. With these conventions, we have two conversion factors:

\[ \frac{180 \text{ deg}}{\pi \text{ rad}} = 1 \quad \frac{\pi \text{ rad}}{180 \text{ deg}} = 1 \]

Example 3.1.2.

(i) Convert 60° to radian measure;
(ii) Convert \( \frac{5\pi}{6} \) to degree measure.

(i) \( 60 \text{ deg} \cdot \frac{\pi \text{ rad}}{180 \text{ deg}} = \frac{\pi}{3} \)

(ii) \( \frac{5\pi}{6} \text{ rad} \cdot \frac{180 \text{ deg}}{\pi \text{ rad}} = 150° \)

Radian measure is rarely used when actually measuring angles in applications in engineering, physics or even more every-day applications such as carpentry. The reason is that it is somewhat clumsy. An angle of measure 1 radian has a degree measure of 57.3°. The real utility of radian measure is that since it is unitless, it converts the trigonometric functions from functions assigning numbers to geometric objects (angles) to ordinary functions on the real numbers.

Figure 3.1.1 shows the boundry angles and the special triangles with angles measured in radians. It would be worth while to become familiar with the radian values of the special angles.
Example 3.1.3.

Using diagrams in Figure 3.1.1,

(i) \( \sin \frac{\pi}{6} = \frac{1}{2} \)

(ii) \( \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \)

(iii) \( \tan \frac{\pi}{3} = \sqrt{3} \)

It is just as easy to evaluate the trigonometric functions on the boundary angles.

Example 3.1.4.

Using the diagram on the left and the definition of the trigonometric functions,

\[
\sin \frac{\pi}{2} = \frac{y}{r} = \frac{1}{1} = 1
\]

\[
\cos \pi = \frac{x}{r} = \frac{-1}{1} = -1
\]

\[
\tan \frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0} \text{ (undefined)}
\]
The procedure for evaluating the trigonometric functions is exactly that same as that for angles measured in degrees.

**Example 3.1.5.**
Evaluate \( \tan \frac{2\pi}{3} \).

\( \frac{2\pi}{3} \) lies in Quad II and has reference angle \( \pi - \frac{2\pi}{3} = \frac{\pi}{3} \).

Using Figure 3.1.1,

\[
\tan \frac{3\pi}{3} = -\tan \frac{\pi}{3} = -\sqrt{3}
\]

**Section 3.1 Problems**

For problems 1–9, convert the angle to radian measure.

1. \( \theta = 120^\circ \).
2. \( \theta = 90^\circ \).
3. \( \theta = 45^\circ \).
4. \( \theta = 210^\circ \).
5. \( \theta = 300^\circ \).
6. \( \theta = 270^\circ \).
7. \( \theta = 135^\circ \).
8. \( \theta = 240^\circ \).
9. \( \theta = 315^\circ \).

For problems 10–15, find three angles coterminal to the given angle.

10. \( \theta = \frac{\pi}{6} \).
11. \( \theta = \frac{3\pi}{4} \).
12. \( \theta = 2\pi \).
13. \( \theta = \frac{5\pi}{4} \).
14. \( \theta = \frac{11\pi}{6} \).
15. \( \theta = \frac{5\pi}{3} \).

For problems 16–24, sketch the angle in standard position and find the reference angle for the given angle.
CHAPTER 3. RADIAN MEASURE

16. \( \theta = \frac{7\pi}{6} \).
19. \( \theta = -\frac{5\pi}{6} \).
22. \( \theta = \frac{5\pi}{3} \).
17. \( \theta = \frac{5\pi}{4} \).
20. \( \theta = -\frac{5\pi}{4} \).
23. \( \theta = \frac{11\pi}{6} \).
18. \( \theta = 2\pi \).
21. \( \theta = \frac{11\pi}{3} \).
24. \( \theta = \frac{5\pi}{3} \).

For problems 22–30, evaluate the trigonometric function.

22. \( \sin \frac{\pi}{6} \).
25. \( \cos \frac{5\pi}{3} \).
28. \( \cos \pi \).
23. \( \tan \frac{3\pi}{4} \).
26. \( \sin \frac{7\pi}{6} \).
29. \( \sin \pi \).
24. \( \theta = 2\pi \).
27. \( \cos \frac{3\pi}{2} \).
30. \( \sin \frac{7\pi}{4} \).
Chapter 4

Graphs of Trigonometric Functions

4.1 The Unit Circle

Let us recall the definition of the sine and cosine functions. If $\theta$ is an angle in standard position and $(x, y)$ is a point on the terminal ray of $\theta$ and $r$ is the distance from $(x, y)$ to the origin, then,

$$\sin \theta = \frac{y}{r} \quad (4.1)$$

$$\cos \theta = \frac{x}{r} \quad (4.2)$$

If $(x, y)$ is the point of intersection of the terminal ray with the unit circle, then, since $r = 1$,

$$\sin \theta = \frac{y}{r} = y$$

$$\cos \theta = \frac{x}{r} = x$$

Figure 4.1.1: The Unit Circle
This is often taken as the definition for the sine and cosine functions. We will state this observation as a theorem.

**Theorem 4.1.1.** If \((x, y)\) is the point of intersection of the terminal ray of \(\theta\) with the unit circle, then \((x, y) = (\cos \theta, \sin \theta)\).

\[
\begin{align*}
\pi/2 & \quad \pi & \quad 0 & \quad 2\pi \\
\theta & \quad r = 1 & \quad (\cos \theta, \sin \theta) & \quad 3\pi/2
\end{align*}
\]

**Example 4.1.2.**

For \(\theta = 5\pi/6\):

(i) Draw the terminal ray of \(\theta\);

(ii) Find the reference angle of \(\theta\);

(iii) Find the coordinates of the point of intersection of \(\theta\) with the unit circle.

Since \(\pi/2 < 5\pi/6 < \pi\), \(\theta\) lies in the second quadrant. Hence \(\theta' = \pi - 5\pi/6 = \pi/6\).

\[
\begin{align*}
\pi/6 & \quad \pi/3 & \quad 2 & \quad \sqrt{3} \\
\theta'/EThetaual\pi/6 & \quad \cos 5\pi/6 = -\sqrt{3}/2 & \quad \sin 5\pi/6 = 1/2
\end{align*}
\]

\((x, y) = (-\sqrt{3}/2, 1/2)\)
Section 4.1 Problems

For each angle given,

(i) Draw the terminal ray of \( \theta \);

(ii) Find the reference angle of \( \theta \);

(iii) Find the coordinates of the point of intersection of \( \theta \) with the unit circle.

1. \( \theta = \frac{\pi}{4} \)
2. \( \theta = \frac{4\pi}{3} \)
3. \( \theta = \frac{11\pi}{6} \)
4. \( \theta = \frac{\pi}{2} \)
5. \( \theta = \pi \)
6. \( \theta = \frac{3\pi}{3} \)

4.2 Graphs of Trigonometric Functions

We now turn to the graphs of the sine and cosine functions. Let \( \theta \) increase from 0 to \( 2\pi \) as in Figure 4.2.1. The \( y \)-coordinate of the point of intersection of the terminal ray of \( \theta \) and the unit circle begins at 0, increases to a maximum value of 1 at \( \theta = \pi/2 \) returns to 0 at \( \theta = \pi \), decreases to a minimum value of \(-1\) at \( \theta = 3\pi/2 \), and then returns back to 0 at \( \theta = 2\pi \). The graph of \( y = \sin \theta \) is then as shown in Figure 4.2.1.

Figure 4.2.1: \( y = \sin \theta \)
Of course increasing $\theta$ above would result in the terminal ray of $\theta$ tracing over the same points on the unit circle several times, so the shape of the sine curve in Figure 4.2.1 will simply repeat itself. Similarly for negative values of $\theta$. We say that the sine function is periodic with period $2\pi$.

A similar analysis of the cosine function yields the following cosine curve.

In general, it suffices to sketch a single period of a periodic curve. Figure 5.1.1 shows the sine and cosine curves for $0 \leq \theta \leq 2\pi$. These curves should be memorized.
Consider now the circle of radius $A$ centered at the origin. If $\theta$ is an angle in standard position and $(x, y)$ is a point on the terminal ray of $\theta$ and $r$ is the distance from $(x, y)$ to the origin, then, by definition,

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}$$

If $(x, y)$ is the point of intersection of the terminal ray with the unit circle, then, since $r = A$,

$$\sin \theta = \frac{y}{A} \Rightarrow y = A \sin \theta$$
$$\cos \theta = \frac{x}{A} \Rightarrow x = A \cos \theta$$

Whereas the minimum and maximum values of $y = \sin \theta$ and $y = \cos \theta$ are $-1$ and $1$, the minimum and maximum values of $y = A \sin \theta$ and $y = A \cos \theta$ are $-A$ and $A$. The graphs retain the same basic shapes, but are stretched vertically.

**Example 4.2.1.**

On the same set of axes, plot:

(i) $y = \sin \theta$ and $y = 2 \sin \theta$
(ii) $y = \cos \theta$ and $y = 2 \cos \theta$
Like with any other function, the graphs of \( y = -\sin \theta \) and \( y = -\cos \theta \) are the graphs of \( y = \sin \theta \) and \( y = \cos \theta \) reflected across the \( x \)-axis.

**Example 4.2.2.**

On the same set of axes, plot:

(i) \( y = -\sin \theta \) and \( y = -2\sin \theta \)

(ii) \( y = -\cos \theta \) and \( y = -2\cos \theta \)

![Graphs](image)

**Figure 4.2.7**

**Definition 4.2.3.** For the functions

\[
\begin{align*}
y & = A \sin \theta \\
y & = A \cos \theta
\end{align*}
\]

(4.3) \hspace{2cm} (4.4)

\(|A|\) is called the amplitude of the function.

**Section 4.2 Problems**

For problems 1-6 graph the given trigonometric function for \( 0 \leq \theta \leq 2\pi \). Label the \( \theta \)-axis with multiples of \( \frac{\pi}{2} \).

1. \( y = 3\sin \theta \)  
2. \( y = -5\sin \theta \)  
3. \( y = 4\cos \theta \)  
4. \( y = 1.5\sin \theta \)  
5. \( y = \frac{1}{2}\cos \theta \)  
6. \( y = -3\cos \theta \)
Chapter 5

Inverse Trigonometric Functions

5.1 Inverse Trigonometric Functions

Let us recall that, in terms of their graphs, functions assign points on the $x$-axis to points on the $y$-axis and that the inverse of a function reverses the process.

Recall also that a function may not have multiple outputs for a single input. This presents a problem for the sine function: its inverse is not a function.
We do wish to define an inverse function for the sine function. To this end we recall that a function has three components: a domain, a range, and a rule that assigns elements of the domain to elements of the range. When defining a function, we have complete control over the first two components.

We now define a new function whose rule agrees with that of the sine function, but we take the domain to be the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. We denote this function $\mathrm{Sin} \theta$.¹ This new (restricted) function has the same range as the sine function (the interval $[-1,1]$), but has the advantage that each element of the range is the image of only one element of the domain.

We denote the inverse of $\mathrm{Sin} \theta$ by $\mathrm{Sin}^{-1} x$.

\[
y = \mathrm{Sin} \theta
\]

![Figure 5.1.3](image-url)

We denote the inverse of $\mathrm{Sin} \theta$ by $\mathrm{Sin}^{-1} x$.

\[
\mathrm{Sin}^{-1} x = \theta \quad \text{if and only if}
\]

(i) $\sin \theta = x$

(ii) $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

We make a similar restriction for the cosine function, except that we cannot use the same interval. We define $\mathrm{Cos} \theta = \cos \theta$ for $0 \leq \theta \leq \pi$.

¹Note this function has a capital $S$ to distinguish it from the ordinary (nonrestricted) sine function.
CHAPTER 4. GRAPHS OF TRIGONOMETRIC FUNCTIONS

Figure 5.1.4

\[
\cos^{-1}x = \theta \quad \text{if and only if} \\
(i) \ \cos\theta = x \\
(ii) \ 0 \leq \theta \leq \pi
\]

For the tangent function, the restriction is similar to that of the sine function.

\[
\tan\theta = \tan\theta \quad \text{for} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}
\]

Figure 5.1.5

\[
\tan^{-1}x = \theta \quad \text{if and only if} \\
(i) \ \tan\theta = x \\
(ii) \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}
\]

Example 5.1.1.

Evaluate \(\cos^{-1}\left(-\frac{1}{2}\right)\).

By the definition of \(\cos^{-1}\), this problem is equivalent to: find \(\theta\) such that \(\cos\theta = -\frac{1}{2}\) and \(0 \leq \theta \leq \pi\).

Since \(\cos\theta = -\frac{1}{2} < 0\), the terminal ray of \(\theta\) lies in either Quad. II or Quad. III.
CHAPTER 4. GRAPHS OF TRIGONOMETRIC FUNCTIONS

Example 5.1.2.

Evaluate $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

By the definition of $\sin^{-1}$, this problem is equivalent to: find $\theta$ such that $\sin\theta = -\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Since $\sin\theta = -\frac{1}{\sqrt{2}} < 0$, the terminal ray of $\theta$ lies in either Quad. III or Quad. IV.

Example 5.1.3.
Evaluate $\tan^{-1}\left(-\frac{5}{4}\right)$. Since $\tan \theta = -\frac{5}{4}$, $\theta$ is not a special angle, so we use a scientific calculator:

$$\tan^{-1}\left(-\frac{5}{4}\right) = -0.8961$$

Section 5.1 Problems

For problems 1-9 evaluate the inverse trigonometric function. Give exact-value answers in radians.

1. $\sin^{-1}\frac{1}{\sqrt{2}}$
2. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
3. $\tan^{-1}1$
4. $\sin^{-1}\left(-\frac{1}{2}\right)$
5. $\cos^{-1}1$
6. $\cos^{-1}\left(-\frac{1}{2}\right)$
7. $\tan^{-1}\left(-\sqrt{3}\right)$
8. $\sin^{-1}1$
9. $\cos^{-1}\left(\frac{1}{2}\right)$

5.2 Trigonometric Equations

The simplest trigonometric is one of the form:

$$\sin \theta = -\frac{1}{2} \quad (5.1)$$

To solve Equation 5.2 means to find all angles $\theta$ that satisfy the equation.

Since $-\frac{1}{2} < 0$, we should expect to find solutions only in Quadrants III and IV, where the sine function is negative. Recall that $\sin \theta' = |\sin \theta|$, where $\theta'$ is the reference angle of $\theta$.

$$\sin \theta = -\frac{1}{2} \Rightarrow |\sin \theta'| = \frac{1}{2}$$

$$\theta' = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Two solutions of Equation 5.2 are:

$$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

$$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

Figure 5.2.1
Any other solution of Equation 5.2 are coterminal with these two, so the general solution is

\[ \theta = \frac{7\pi}{6} + 2k\pi \]
\[ \theta = \frac{11\pi}{6} + 2k\pi \]

\[ k = 0, \pm 1, \pm 2, \pm 3, \ldots \]

In general, it suffices to find all solutions \( \theta \) of a trigonometric equation that satisfy \( 0 \leq \theta < 2\pi \). Any other solution will be coterminal with one of these.

In most cases, some algebra is involved to put a trigonometric equation in the form of Equation 5.2.

**Example 5.2.1.**

Find all solutions to

\[ \sqrt{2}\cos \theta - 1 = 0 \]

that satisfy \( 0 \leq \theta < 2\pi \).

We first solve algebraically for \( \cos \theta \):

\[ \sqrt{2}\cos \theta - 1 = 0 \]
\[ \sqrt{2}\cos \theta = 1 \]
\[ \cos \theta = \frac{1}{\sqrt{2}} \]

Since \( \cos \theta = \frac{1}{\sqrt{2}} > 0 \), we will find solutions only in Quadrants I and IV, where the cosine function is positive.

The two solutions of the equation are:

\[ \theta = \frac{\pi}{4} \]
\[ \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4} \]

Figure 5.2.2
Of course, the solutions to trigonometric equations are not always special angles.

**Example 5.2.2.**

Find all solutions to 

$$2 \tan \theta + 3 = 0$$

that satisfy \(0 \leq \theta < 2\pi\).

We first solve algebraically for \(\cos \theta\):

\[
2 \tan \theta + 3 = 0 \\
2 \tan \theta = -3 \\
\tan \theta = -\frac{3}{2}
\]

Since \(\tan \theta = -\frac{3}{2} < 0\), we will find solutions only in Quadrants II and IV, where the tangent function is negative.

\[
\tan \theta = -\frac{3}{2} \\
\tan \theta' = \frac{3}{2} \\
\theta' = \tan^{-1}\left(\frac{3}{2}\right) = 0.9828
\]

The two solutions of the equation are:

\[
\theta = 2\pi - \theta' = 5.3004 \\
\theta = \pi - \theta' = 2.1588
\]

**Figure 5.2.3**

---

**Section 5.2 Problems**

For problems 1-9 find all solutions for the trigonometric equation satisfying \(0 \leq \theta < 2\pi\). Where possible, give exact-value answers. (This is when the solutions are special angles.)

1. \(2 \sin \theta - 1 = 0\)  
2. \(2 \cos \theta + \sqrt{3} = 0\)  
3. \(\tan \theta - 1 = 0\)  
4. \(1.5 \sin \theta + 2.3 = 0\)  
5. \(3 \cos \theta - 2 = 0\)  
6. \(5 \tan \theta + 4 = 0\)  
7. \(4 \cos \theta - 2 = 0\)  
8. \(\sqrt{2} \sin \theta - 1 = 0\)  
9. \(\tan \theta - \sqrt{3} = 0\)
Final Review

Review

1. Convert the following angles to radian measure. Give exact-value answers.

(a) 15°
(b) −225°
(c) 330°
(d) 270°
(e) 60°
(f) 120°

2. Convert the following angles to degree measure.

(a) $\frac{3\pi}{8}$
(b) 1.5
(c) 5.25
(d) $\frac{\pi}{6}$
(e) −2.7
(f) $\frac{4\pi}{3}$

3. A 16-foot ladder is resting against a wall. The top of the ladder is 15 feet from the ground. What angle does the ladder make with the wall?

4. From a point 200 feet from the base of a Roman aqueduct in northern Spain the angle of elevation to the top of the aqueduct is $19^\circ$. Find the height of the aqueduct.

5. Find the values of the six trigonometric functions on an angle whose terminal ray passes through the point $P(7, -4)$.

6. Find the exact value of $\sin \theta$, $\cos \theta$, and $\tan \theta$ for the following angles.

(a) $\theta = \pi$
(b) $\theta = \frac{3\pi}{4}$
(c) $\theta = \frac{5\pi}{6}$
(d) $\theta = \frac{3\pi}{2}$
(e) $\theta = \frac{7\pi}{4}$
(f) $\theta = \frac{4\pi}{3}$

7. Use a calculator to evaluate the following expressions. Give your answer accurate to 4 decimal places.

(a) $\sin 127^\circ$
(b) $\cos(−116^\circ)$
(c) $\sec(−4.45)$
(d) $\csc 0.34$

8. Find two values $\theta$ with $0 \leq \theta < 2\pi$ that satisfy the trigonometric equation. Give exact-value answers.

(a) $\sin \theta = \frac{1}{2}$
(b) $\cos \theta = -\frac{\sqrt{3}}{2}$
(c) $\tan \theta = -\frac{1}{\sqrt{3}}$
(d) $\tan \theta = 1$

9. Graph the following functions. State the domain, range, amplitude and period.

(a) $y = 2\sin \theta$
(b) $y = \frac{1}{2} \cos \theta$

10. Solve the following equations in the interval $0 \leq \theta < 2\pi$.

(a) $2\sin \theta - \sqrt{2} = 0$
(b) $\cos \theta = 0.6725$
(c) $2\cos \theta + 1 = 0$
(d) $2\tan \theta + 5 = 0$

11. Use the definitions of $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, and $\tan \theta = \frac{y}{x}$ to prove the following trigonometric identities.

(a) $\sin^2 \theta + \cos^2 \theta = 1$
(b) $\tan \theta = \frac{\sin \theta}{\cos \theta}$
(c) $1 + \tan^2 \theta = \sec^2 \theta$
(d) $1 + \cot^2 \theta = \sec^2 \theta$

12. Multiply and simplify:

(a) $(\sin \theta - \cos \theta)(\sin \theta + \cos \theta)$
(b) $(\sin \theta - \cos \theta)^2$
(c) $(1 + \tan \theta)^2$
(d) $\tan \theta(\cos \theta - \csc \theta)$
CHAPTER 4. GRAPHS OF TRIGONOMETRIC FUNCTIONS

Practice Test

1. Convert $36^\circ$ to radian measure. Give an exact-value answer.

2. Convert $\frac{5\pi}{8}$ to degree measure. Give your answer accurate to two decimal places.

3. At a point 180 feet away from the base of a building, the angle of elevation to the top of the building is $38^\circ$. What is the height of the building? Give your answer correct to two decimal places.

4. Use a calculator to evaluate the following. Give your answers correct to four decimal places. (Be sure to check your mode.)
   
   (a) $\sin 72^\circ$
   (b) $\cos(-3.4)$

5. Evaluate the following. Give exact-value answers.
   
   (a) $\sin \frac{\pi}{3}$
   (b) $\cos \frac{2\pi}{3}$
   (c) $\tan \frac{5\pi}{4}$
   (d) $\sin \frac{\pi}{2}$
   (e) $\tan \pi$
   (f) $\cos \frac{11\pi}{6}$

6. Find the exact value in radians of
   
   $\cos^{-1} \left( -\frac{1}{2} \right)$

7. Find the exact value in radians of
   
   $\tan^{-1}(1)$

8. Sketch the graph of
   
   $y = 3\sin \theta$
   
   for $0 \leq \theta \leq 2\pi$. Label the $\theta$-axis with multiples of $\frac{\pi}{2}$ and state the amplitude of the function.

9. Multiply and simplify fully:
   
   $\sin \theta \left( \cot \theta + \tan \theta \right)$

10. Find the value of the six trigonometric functions on the angle shown in the diagram below.
Appendix A

The Greek Alphabet

<table>
<thead>
<tr>
<th>Greek</th>
<th>English</th>
</tr>
</thead>
<tbody>
<tr>
<td>α A</td>
<td>ν N</td>
</tr>
<tr>
<td>β B</td>
<td>xi ξ Ξ</td>
</tr>
<tr>
<td>γ Γ</td>
<td>omicron o O</td>
</tr>
<tr>
<td>δ Δ</td>
<td>pi π Π</td>
</tr>
<tr>
<td>ε E</td>
<td>rho ρ P</td>
</tr>
<tr>
<td>ζ Z</td>
<td>sigma σ Σ</td>
</tr>
<tr>
<td>η E</td>
<td>tau τ T</td>
</tr>
<tr>
<td>θ Θ</td>
<td>upsilon υ Υ</td>
</tr>
<tr>
<td>ι I</td>
<td>phi φ Φ</td>
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<tr>
<td>κ K</td>
<td>chi χ X</td>
</tr>
<tr>
<td>λ Λ</td>
<td>psi ψ Ψ</td>
</tr>
<tr>
<td>μ M</td>
<td>omega ω Ω</td>
</tr>
</tbody>
</table>
Appendix B

Midcourse Review Answers

1. Find the values of all six trigonometric functions on angle A in the triangle below.

   ![Triangle with sides 6 and 3.5, angle 33°]

   Using Pythagoras, \( b = 5 \).

   \[
   \begin{array}{ccc}
   \sin A &=& \frac{\sqrt{11}}{6} \\
   \cos A &=& \frac{5}{6} \\
   \tan A &=& \frac{\sqrt{11}}{5}
   \end{array}
   \]

   \[
   \begin{array}{ccc}
   \csc A &=& \frac{6}{\sqrt{11}} \\
   \sec A &=& \frac{6}{5} \\
   \cot A &=& \frac{5}{\sqrt{11}}
   \end{array}
   \]

2. Find the reference angles for each of the following angles.

   (a) 60°
   (b) 32°
   (c) 10°
   (d) 83°
   (e) 30°
   (f) 45°

3. Given that \( \alpha \) is an acute angle and \( \sin \alpha = \frac{2}{3} \), find the values of the remaining 5 trigonometric functions on \( \theta \).

   Using Pythagoras, the remaining side is \( \sqrt{5} \).

   \[
   \begin{array}{ccc}
   \sin \theta &=& \frac{2}{\sqrt{5}} \\
   \cos \theta &=& \frac{\sqrt{5}}{3} \\
   \tan \theta &=& \frac{2}{\sqrt{5}}
   \end{array}
   \]

   \[
   \begin{array}{ccc}
   \csc \theta &=& \frac{3}{2} \\
   \sec \theta &=& \frac{\sqrt{5}}{3} \\
   \cot \theta &=& \frac{\sqrt{5}}{2}
   \end{array}
   \]

4. Given that 90° < \( \theta < 180° \) and \( \sin \theta = 2/3 \), find the values of the remaining 5 trigonometric functions on \( \theta \).

   \[
   \begin{array}{ccc}
   \sin \theta &=& 2/3 \\
   \cos \theta &=& -\sqrt{5}/3 \\
   \tan \theta &=& -2/\sqrt{5}
   \end{array}
   \]

   from problem 3. Since \( \theta \) lies in Quad IV,

   \[
   \begin{array}{ccc}
   \sin \theta &=& \frac{2}{3} \\
   \cos \theta &=& -\frac{\sqrt{5}}{3} \\
   \tan \theta &=& -\frac{2}{\sqrt{5}}
   \end{array}
   \]

   \[
   \begin{array}{ccc}
   \csc \theta &=& \frac{3}{2} \\
   \sec \theta &=& -\frac{\sqrt{5}}{3} \\
   \cot \theta &=& -\frac{2}{\sqrt{5}}
   \end{array}
   \]

5. Solve the right triangle in the figure below.

   \[
   B = 90 - 33° = 57°
   \]

   \[
   \begin{array}{ccc}
   \tan 33° &=& \frac{a}{3.5} \\
   a &=& 3.5 \tan 33° = 2.3
   \end{array}
   \]

   Using Pythagoras,

   \[
   c = \sqrt{3.5^2 + 2.3^2} = 4.2
   \]

6. Evaluate \( \sin^2 23° + \cos^2 23° \).

   \[
   \sin^2 23° + \cos^2 23° = 1
   \]

7. Given that \( \sin 23° = 0.3907 \), find:

   Note that 90° - 23° = 67°, so \( \cos 67° = \sin 23° \).

   After finding the reference angle for each angle below, and considering the quadrant in which the angle lies, we find:

   (a) \( \sin 157° = .3907 \)
   (b) \( \sin 203° = -.3907 \)
   (c) \( \sin 337° = -.3907 \)
   (d) \( \cos 67° = .3907 \)
   (e) \( \cos 113° = -.3907 \)
   (f) \( \cos(-113°) = -.3907 \)

8. Use the special triangles to evaluate:

   (a) \( \cos 120° = -\frac{1}{2} \)
   (b) \( \tan 210° = \frac{1}{\sqrt{3}} \)
   (c) \( \sin 330° = -\frac{1}{2} \)
   (d) \( \cos 135° = -\frac{1}{\sqrt{2}} \)
   (e) \( \tan(-60°) = -\sqrt{3} \)
   (f) \( \sin 150° = \frac{1}{2} \)

9. Evaluate:

   (a) \( \cos 90° = 0 \)
   (b) \( \tan 180° = 0 \)
   (c) \( \sin 0° = 0 \)
   (d) \( \sin 270° = -1 \)
   (e) \( \tan 270°: \text{undefined} \)
   (f) \( \cos 0° = 1 \)

(See Figure B.0.1)
10. Given $\sin \theta = -0.225$.

(a) Find $\theta'$, the reference angle for $\theta$.

\[ \sin^{-1}(-0.225) = 13^\circ \]

(b) Given that $\theta$ lies in Quad. IV, evaluate:

i. $\cos \theta = \cos 13^\circ = 0.974$

ii. $\tan \theta = \tan 13^\circ = -0.231$

11. Solve the right triangle in the figure below.

Using Pythagoras,

\[ c = \sqrt{3.1^2 + 4.7^2} = 5.6 \]

\[ A = \tan^{-1} \left( \frac{3.1}{4.7} \right) = 33.4^\circ \]

\[ B = 90^\circ = 33.4^\circ = 56.6^\circ \]

(d) \[ \frac{1}{1 - \sin \theta} - \frac{1}{1 + \sin \theta} = 2 \tan \theta \sec \theta \]

\[ \frac{1}{1 - \sin \theta} - \frac{1}{1 + \sin \theta} = \frac{1}{1 - \sin \theta} \frac{1 + \sin \theta}{1 + \sin \theta} - \frac{1}{1 + \sin \theta} \frac{1 + \sin \theta}{1 - \sin \theta} \]

\[ = \frac{1 + \sin \theta - (1 - \sin \theta)}{1 - \sin^2 \theta} = \frac{2 \sin \theta}{\cos^2 \theta} = \frac{2 \sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta} = 2 \tan \theta \sec \theta \]

12. Prove the following trigonometric identities:

(a) \[ \frac{\cot \theta}{\csc \theta} = \cos \theta \]

\[ \cot \theta = \frac{\cos \theta}{\sin \theta} \]

\[ \csc \theta = \frac{1}{\sin \theta} \]

\[ \frac{\cot \theta}{\csc \theta} = \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\sin \theta} = \cos \theta \]

(b) \[ \cot \alpha + \tan \alpha = \sec \alpha \csc \alpha \]

\[ \cot \alpha + \tan \alpha = \frac{\cos \alpha}{\sin \alpha} + \frac{\sin \alpha}{\cos \alpha} \]

\[ = \frac{\cos^2 \alpha + \sin^2 \alpha}{\sin \alpha \cos \alpha} \]

\[ = \frac{1}{\sin \alpha \cos \alpha} \]

\[ = \frac{1}{\sin \alpha} \cdot \frac{1}{\cos \alpha} \]

\[ = \sec \alpha \csc \alpha \]

(c) \[ \tan \beta \sin \beta + \cos \beta = \sec \beta \]

\[ \tan \beta \sin \beta + \cos \beta = \frac{\sin \beta}{\cos \beta} \sin \beta + \cos \beta \]

\[ = \frac{\sin^2 \beta + \cos \beta}{\cos \beta} \]

\[ = \frac{\sin^2 \beta + \cos^2 \beta}{\cos \beta} \]

\[ = \frac{\sin^2 \beta + \cos^2 \beta}{\cos \beta} \]

\[ = \frac{1}{\cos \beta} = \sec \beta \]

(d) \[ \frac{1}{1 - \sin \theta} - \frac{1}{1 + \sin \theta} = 2 \tan \theta \sec \theta \]
Appendix C

Final Review Answers

1. Convert the following angles to radian measure.
   Give exact-value answers.
   
   (a) $\frac{\pi}{12}$ 
   (b) $-\frac{5\pi}{4}$ 
   (c) $\frac{11\pi}{6}$ 
   (d) $\frac{3\pi}{2}$ 
   (e) $\frac{\pi}{3}$ 
   (f) $\frac{2\pi}{3}$ 

2. Convert the following angles to degree measure.
   
   (a) $67.5^\circ$ 
   (b) $85.9^\circ$ 
   (c) $300.8^\circ$ 
   (d) $30^\circ$ 
   (e) $-154.7^\circ$ 
   (f) $240^\circ$ 

3. $\cos \theta = \frac{15}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(\frac{15}{\sqrt{2}}\right) = 20.4^\circ$

4. $\tan 19^\circ = \frac{h}{200} \Rightarrow h = 200 \tan 19^\circ = 68.9$ ft

5. $x = 7, y = -4$. $r^2 = 7^2 + (-4)^2 = \sqrt{65}$
   
   (a) $\sin \theta = -\frac{4}{\sqrt{65}}$ 
   (b) $\cos \theta = \frac{7}{\sqrt{65}}$ 
   (c) $\tan \theta = -\frac{4}{7}$ 
   (d) $\csc \theta = -\frac{\sqrt{65}}{4}$ 
   (e) $\sec \theta = \frac{\sqrt{65}}{7}$ 
   (f) $\cos \theta = -\frac{7}{4}$

6. Find the exact value of $\sin \theta$, $\cos \theta$, and $\tan \theta$ for the following angles.

   (a) $\sin \pi = 0$ 
   (b) $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$ 
   (c) $\sin \frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$
   (d) $\cos \pi = -1$ 
   (e) $\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ 
   (f) $\cos \frac{4\pi}{3} = -\frac{1}{2}$
   (g) $\tan \pi = 0$ 
   (h) $\tan \frac{3\pi}{4} = -1$ 
   (i) $\tan \frac{4\pi}{3} = -\sqrt{3}$
   (j) $\tan \frac{7\pi}{4} = 1$ 
   (k) $\tan \frac{3\pi}{2}$: undef.

7. Use a calculator to evaluate the following expressions. Give your answer accurate to 4 decimal places.
   
   (a) 0.7986 
   (b) -0.4384 
   (c) -3.8552 
   (d) 2.9986

8. Find two values $\theta$ with $0 \leq \theta < 2\pi$ that satisfy the trigonometric equation. Give exact-value answers.
   
   (a) $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ 
   (b) $\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$ 
   (c) $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$ 
   (d) $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ 
   (e) $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ 
   (f) $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

9. Graph the following functions. State the domain, range, amplitude and period.
amplitude: 2; period: 2π.

10. Solve the following equations in the interval 0 ≤ θ < 2π.
   (a) θ = π/4, 3π/4
   (b) θ = 0.8332, 5.4500
   (c) θ = 2π/3, 4π/3
   (d) θ = 1.9513, 5.0929

11. Use the definitions of sin θ = y/r, cos θ = x/r, and tan θ = y/x to prove the following trigonometric identities.
   (a) 
   \[ \sin^2 \theta + \cos^2 \theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1 \]
   (b) 
   \[ \frac{\sin \theta}{\cos \theta} = \frac{y/r}{x/r} = \frac{y}{x} = \frac{\tan \theta}{1} \]

12. Multiply and simplify:
   (a) (\sin \theta - \cos \theta)(\sin \theta + \cos \theta) = \sin^2 \theta - \cos^2 \theta
   (b) 
   \[ (\sin \theta - \cos \theta)^2 = (\sin \theta - \cos \theta)(\sin \theta - \cos \theta) = \sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta = 1 - 2 \sin \theta \cos \theta \]
   (c) 
   \[ (1 + \tan \theta)^2 = (1 + \tan \theta)(1 + \tan \theta) = 1 + 2 \tan \theta + \tan^2 \theta = \sec^2 \theta + 2 \tan \theta \]
   (d) 
   \[ \tan \theta(\cos \theta - \csc \theta) = \frac{\sin \theta}{\cos \theta} \left(\cos \theta - \frac{1}{\sin \theta}\right) = \sin \theta - \frac{1}{\cos \theta} = \sin \theta - \sec \theta \]
Appendix D

Answers to Odd-Numbered Problems

Section 1.1
1. \( \frac{2}{3} \)
3. \( \frac{5}{2} \)
5. \( \sqrt{13} \)
7. \( \sqrt{5} \)
9. 102°
11. 40°
13. 63°

Section 1.2
1. \( B = 68° \)
\( b = 21.5 \)
\( c = 23.2 \)
3. \( A = 38° \)
\( a = 5.0 \)
\( b = 6.4 \)
5. \( A = 34.5° \)
\( B = 55.5° \)
\( c = 7.6 \)
7. \( A = 43° \)
\( a = 7.4 \)
\( c = 10.8 \)

9. \( A = 37° \)
\( a = 3.2 \)
\( b = 4.5 \)
\( \sin \theta = \frac{12}{13} \quad \csc \theta = \frac{13}{12} \)
\( \cos \theta = \frac{5}{13} \quad \sec \theta = \frac{13}{5} \)
\( \tan \theta = \frac{12}{5} \quad \cot \theta = \frac{5}{12} \)
\( \sin \theta = \frac{1}{2} \quad \csc \theta = 2 \)
\( \cos \theta = \frac{\sqrt{3}}{2} \quad \sec \theta = \frac{2}{\sqrt{3}} \)
\( \tan \theta = \frac{1}{\sqrt{3}} \quad \cot \theta = \sqrt{3} \)
\( \sin \theta = \frac{\sqrt{3}}{2} \quad \csc \theta = \frac{2}{\sqrt{3}} \)
\( \cos \theta = \frac{1}{2} \quad \sec \theta = 2 \)
\( \tan \theta = \sqrt{3} \quad \cot \theta = \frac{1}{\sqrt{3}} \)
17. 24.1 ft.
19. 33.0°
21. 77.9 ft
23. 13.5 ft

Section 1.3
1. 60°
3. 65°
5. 27°

7. 45°

9. 50°

11. 560°, 920°, −160°.

13. 140°, 500°, −580°.

15. 670°, 1030°, −50°.

17. 
\[ \sin \theta = -\frac{4}{5} \quad \csc \theta = -\frac{5}{4} \]
\[ \cos \theta = -\frac{3}{5} \quad \sec \theta = -\frac{5}{3} \]
\[ \tan \theta = \frac{4}{3} \quad \cot \theta = \frac{3}{4} \]

19. 
\[ \sin \theta = 0 \quad \csc \theta : \text{undef.} \]
\[ \cos \theta = -1 \quad \sec \theta = -1 \]
\[ \tan \theta = 0 \quad \cot \theta : \text{undef.} \]

21. 

Section 2.1

1. 
\[ \tan \theta = \frac{\sin \theta}{\cos \theta} \]
\[ = \text{sin} \theta / \text{cos} \theta \]
\[ = \sin \theta / \cos \theta \]
\[ = \text{sec} \theta \]

3. 
\[ \sin \theta + \sin \theta \cot^2 \theta = \sin \theta(1 + \cot^2 \theta) \]
\[ = \sin \theta \csc^2 \theta \]
\[ = \sin \theta \frac{1}{\sin^2 \theta} \]
\[ = \frac{1}{\sin \theta} \]
\[ = \csc \theta \]
9. \[
\frac{\sec \theta - 1}{\sec^2 \theta + 1} = \frac{\frac{1}{\cos \theta} - 1}{1 + \frac{1}{\cos \theta}} = \frac{\frac{1 - \cos \theta}{\cos \theta}}{\frac{1 + \cos \theta}{\cos \theta}} = \frac{1 - \cos \theta}{1 + \cos \theta} \cdot \frac{\cos \theta}{\cos \theta} = 1 - \cos \theta.
\]

13. 225°

15. 300°

17. \( \theta' = \frac{\pi}{4} \)

19. \( \theta' = \frac{\pi}{6} \)

21. \( \theta' = \frac{\pi}{3} \)

23. -1

25. \( \frac{1}{2} \)

Section 3.1

1. \( \frac{2\pi}{3} \)

3. \( \frac{\pi}{4} \)

5. \( \frac{5\pi}{4} \)

7. \( \frac{3\pi}{4} \)

9. \( \frac{7\pi}{4} \)

11. 135°

Section 4.1

1. \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \)

3. \( \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \)

5. \( (-1, 0) \)

Section 4.2
Section 5.1
1. $\frac{\pi}{4}$
3. $\theta = \frac{\pi}{4}$
5. $0$
7. $-\frac{\pi}{3}$
9. $\frac{\pi}{3}$

3. $\theta = \frac{5\pi}{4}$
5. $\theta = 0.8411$
7. $\theta = \frac{\pi}{3}$
9. $\theta = \frac{5\pi}{3}$

Section 5.2
1. $\theta = \frac{\pi}{6}$
9. $\theta = \frac{\pi}{3}$

$\theta = \frac{5\pi}{6}$
$\theta = \frac{4\pi}{3}$
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